Creating a Cluster with Keplerian Orbits

M. Atakan Gürkan, *UvA*

05 June 2007

1 Introduction

Given a mass/density distribution, an eccentricity distribution and the mass of the central object, we want to make a cluster of stars on Keplerian orbits. We neglect their interaction with one another for constructing the orbits.

The formulae below are taken from a MATLAB guide entitled "Orbital Mechanics with MATLAB", I found online. They are also available in "Orbital Mechanics, Second Edition" (V. A. Chobotov, ed., 1996) section 4.8, but this book is not easy to find.

2 Procedure

Orbits are most easily constructed using orbital elements and then transforming to Cartesian coordinates. Here are the orbital elements:

- a: semimajor axis, set by using mass/density distribution;
- e: eccentricity, set by using eccentricity distribution;
- I: inclination, $\cos I$ chosen uniformly from $[-1, 1]$;
- Ω: longitude of the ascending node, chosen uniformly from $[0, 2\pi)$;
- ω: argument of the periastron, chosen uniformly from $[0, 2\pi)$;
- $\mathcal{M}$: mean anomaly (or equivalently, $\mathcal{E}$: eccentric anomaly; or equivalently, ν: true anomaly; or equivalently, τ: time of pericentre passage), chosen uniformly from $[0, 2\pi)$.

See Figure 1. The first two describe the shape of the orbit. The next two describe the orientation of the orbital plane. The fifth sets the orientation of the orbit in the orbital plane and the sixth one sets the position of the object on the orbit.

2.1 Calculating the Orbital Elements

If we know the distribution function $g(x)$ for a quantity, we can sample from that distribution function by solving the equation

$$\int_{x_{\min}}^{x} g(y)dy = X \int_{x_{\min}}^{x_{\max}} g(y)dy \,, \tag{1}$$

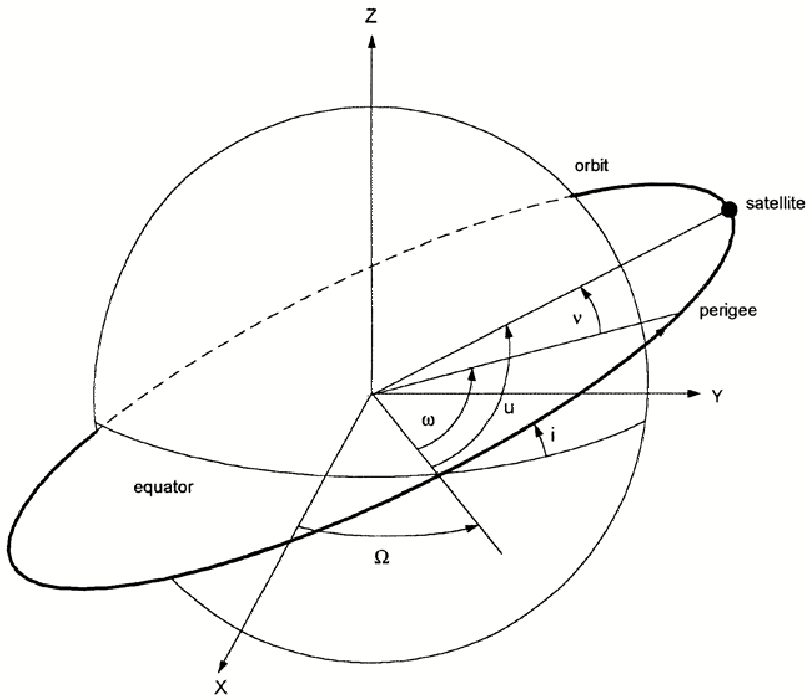

Figure 1: Orbital elements (from MATLAB documentation)

for x, where X is random number in interval $[0, 1]$. For example if the eccentricity is given by a thermal distribution:

$$g(e) = 2e\,, \tag{2}$$

then a random eccentricity can be found simply by $e(X) = \sqrt{X}$.

To find the semimajor axis we need either a density or cumulative mass distribution. We can convert from density to cumulative mass distribution by integrating

$$M_<(r) = 4\pi \int_0^{r_{\max}} \rho(r) r^2 dr\,. \tag{3}$$

Then we can find a by solving

$$M_<(a) = X M_<(a_{\max})\,, \tag{4}$$

where X is again a random number.

[I am not sure of the following – a.] Actually we need to use the distribution function for $1/r$ and obtain $1/a$ by sampling from it, since in a Keplerian orbit the time average of $1/r$ is $1/a$, but the time average of r is *not* a. However, I am too lazy to do this, the density law is usually chosen arbitrarily and this treatment gives identical result for a pure powerlaw cusp anyway.

We need to find eccentric anomaly $\mathcal{E}$ and true anomaly ν from mean anomaly $\mathcal{M}$ and eccentricity e to be able to use the formulae in the next section. This is done by solving the Kepler equation

$$\mathcal{M} = \mathcal{E} - e \sin \mathcal{E}\,, \tag{5}$$

and using

$$\tan \frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{\mathcal{E}}{2}\,. \tag{6}$$

2.2 Converting from Orbital Elements to Cartesian Coordinates

We can calculate the distance from force centre r, and velocity v using

$$r = a(1 - e \cos \mathcal{E})\,, \tag{7}$$

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right)\,, \tag{8}$$

where $\mu = GM$ and M is the mass of the central object.

We calculate the components by :

$$r_x = r[\cos\Omega\cos(\omega+\nu) - \sin\Omega\cos I\sin(\omega+\nu)] \tag{9}$$
$$r_y = r[\sin\Omega\cos(\omega+\nu) + \cos\Omega\cos I\sin(\omega+\nu)] \tag{10}$$
$$r_z = r\sin I\sin(\omega+\nu) \tag{11}$$
$$v_x = -\left(\frac{\mu}{p}\right)^{1/2}[\cos\Omega\{\sin(\omega+\nu)+e\sin\omega\} - \sin\Omega\cos I\{\cos(\omega+\nu)+e\cos\omega\}] \tag{12}$$
$$v_y = -\left(\frac{\mu}{p}\right)^{1/2}[\sin\Omega\{\sin(\omega+\nu)+e\sin\omega\} - \cos\Omega\cos I\{\cos(\omega+\nu)+e\cos\omega\}] \tag{13}$$
$$v_z = \left(\frac{\mu}{p}\right)^{1/2}\sin I\{\cos(\omega+\nu)+e\cos\omega\} \tag{14}$$

where

$$p = a(1-e^2)\,. \tag{15}$$

There is not always a one-to-one correspondence between orbital elements and Cartesian position and velocities. The degenerate cases needs to be handled seperately.

2.2.1 Circular orbit

When $e = 0$, ω becomes meaningless, since there is no periastron. In this case the only quantity that appears is $\omega + \nu$; so, rather than sampling $\mathcal{M}$ and ω seperately we sample this sum from a uniform distribution in $[0, 2\pi)$.

2.2.2 Zero inclination orbit

If $I = 0$ then the angular momentum of the orbit is parallel to z-axis. In this case, Ω becomes meaningless, but we still assign a value to it.