Parallel Graph Algorithms (Chapter 10)

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Schedule for rest of semester

- **4/11/08 - Deadline for finalizing Assignment #4**
  - **Send email to TA and me by tomorrow with your choices for**
    - **Parallel Language**
    - **Parallel Hardware**
    - **Sequential program that you'd like to parallelize**

      It can be the same as one of the previous assignments, but now rewritten in a different parallel language
- **4/22/08 - In-class final exam**
- **4/30/08 - Deadline for Assignment #4**

Acknowledgments for today's lecture

- Slides accompanying course textbook
  - http://www-users.cs.umn.edu/~karypis/parbook/
- John Mellor-Crummey --- COMP 422 slides from Spring 2007

Topics for Today

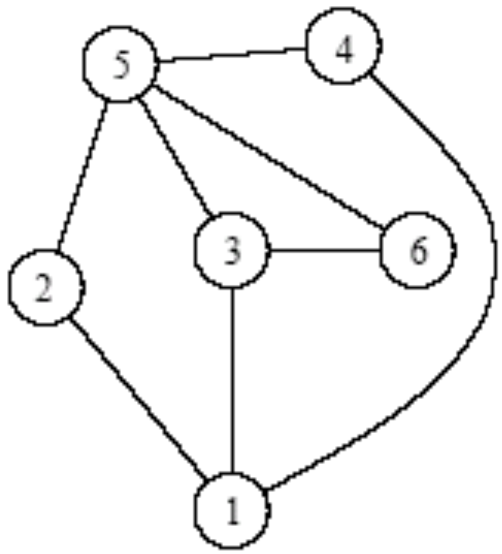
- Terminology and graph representations
- Minimum spanning tree, Prim's algorithm
- Shortest path, Dijkstra’s algorithm, Johnson’s algorithm
- Connected components

Terminology

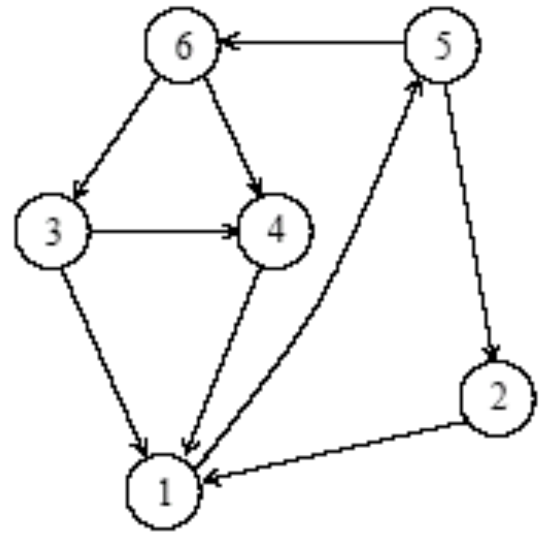
- ***Graph*** **$G = (V,E)$**
  - —**V is a finite set of points called *vertices***
  - —**E is a finite set of *edges***
- ***Undirected graph***
  - —***edge*** **$e \in E$**
    - – **unordered pair (u,v), where $u,v \in V$**
- **Directed graph**
  - —**edge $(u,v) \in E$**
    - – ***incident from*** **vertex u**
    - – ***incident to*** **vertex v**
- ***Path*** **from a vertex u to v**
  - – **a sequence $<v_0,v_1,v_2,\ldots,v_k>$ of vertices**

    $v_0 = u$, $v_k = v$, and $(v_i, v_{i+1}) \in E$ for $i = 0, 1,\ldots, k-1$
  - —**path length = # of edges in a path**

Directed and Undirected Graph Examples



undirected graph



directed graph

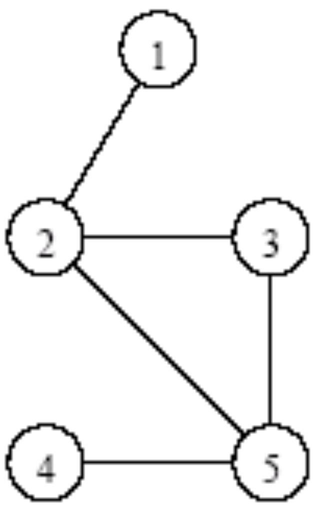
More Terminology

- ***Connected*** **undirected graph**
  - **—every pair of vertices is connected by a path.**
- ***Forest:*** **an acyclic graph**
- ***Tree:*** **a connected acyclic graph**
- ***Weighted graph*****: graph with edge weights**

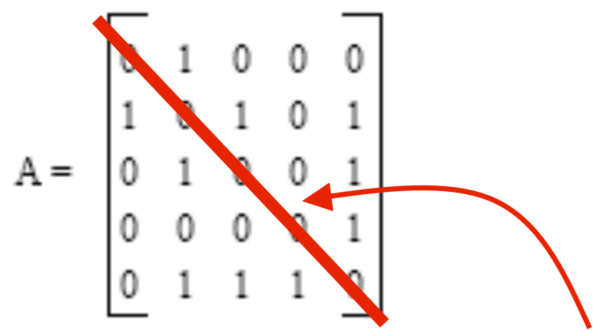
- **Common graph representations**
  - **—adjacency matrix**
  - **—adjacency list**

Adjacency Matrix for Graph G = (V,E)

- **|V| x |V| matrix**
  - **—matrix element $a_{i,j}$ = *1* if nodes *i* and *j* share an edge; 0 otherwise**
  - **—for a weighted graph, $a_{i,j} = w_{i,j}$, the edge weight**
- **Requires $\Theta(|V|^2)$ space**



Undirected graph

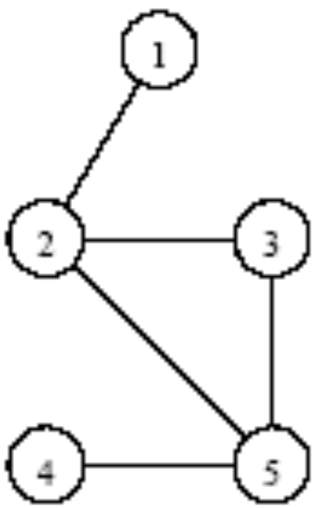


Adjacency matrix representation

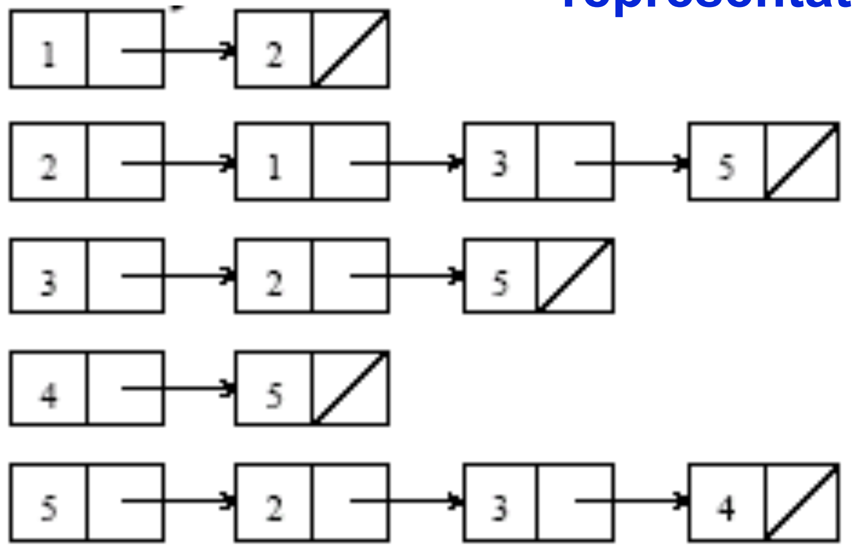
adjacency matrix is symmetric about the diagonal

Adjacency List for Graph G = (V,E)

- **An array *Adj[1..*|V|*]* of lists**
  - **—each list *Adj[v]* is a list of all vertices adjacent to *v***
- **Requires** $\Theta(|E|)$ **space**



Adjacency list representation



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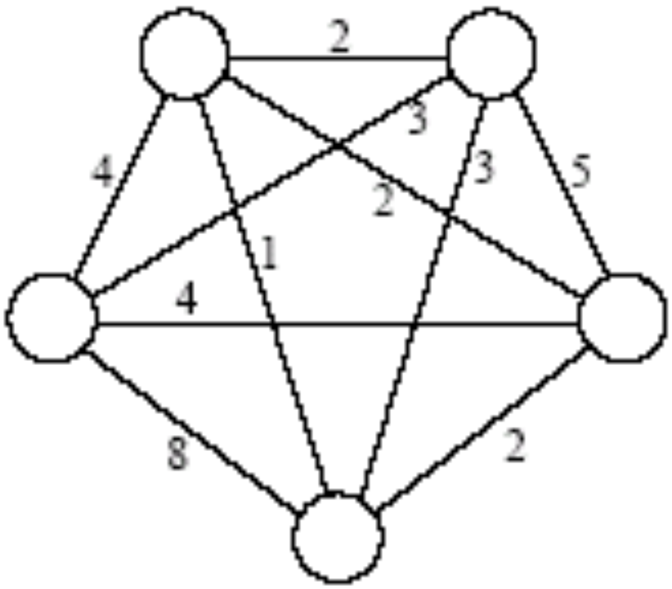
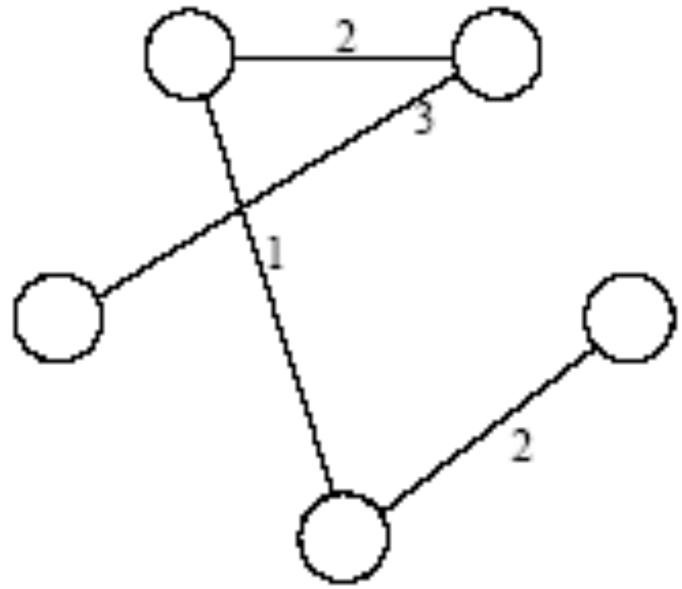
Minimum Spanning Tree

- ***Spanning tree*** **of a connected undirected graph *G***
  - **—subgraph of *G* that is a tree containing all the vertices of *G***
  - **—if graph is not connected: spanning forest**
- **Weight of a subgraph in a weighted graph**
  - **— sum of the weights of the edges in the subgraph**
- ***Minimum spanning tree*** **(MST) for weighted undirected graph**
  - **—spanning tree with minimum weight**

Minimum Spanning Tree



Undirected graph



Minimum spanning tree

Computing a Minimum Spanning Tree

Prim's sequential algorithm

procedure PRIM_MST(V, E, w, r)
begin
  $V_T := \{r\}$; // initialize spanning tree vertices V_T with vertex r, the designated root
  $d[r] := 0$;
  for all $v \in (V - V_T)$ **do**
    if edge (r, v) exists set $d[v] := w(r, v)$;
    else set $d[v] := \infty$;
  // compute d[:], the
  // weight between
  // r and each
  // vertex outside V_T
  while $V_T \neq V$ **do** // while there are vertices outside T
  begin
    find a vertex u such that $d[u] := \min\{d[v] | v \in (V - V_T)\}$;
    $V_T := V_T \cup \{u\}$; // add u to T
    for all $v \in (V - V_T)$ **do**
      $d[v] := \min\{d[v], w(u, v)\}$; // recompute d[:] now
      // that u is in T
  endwhile
end PRIM_MST

// use d[:] to find u,
// vertex closest to T

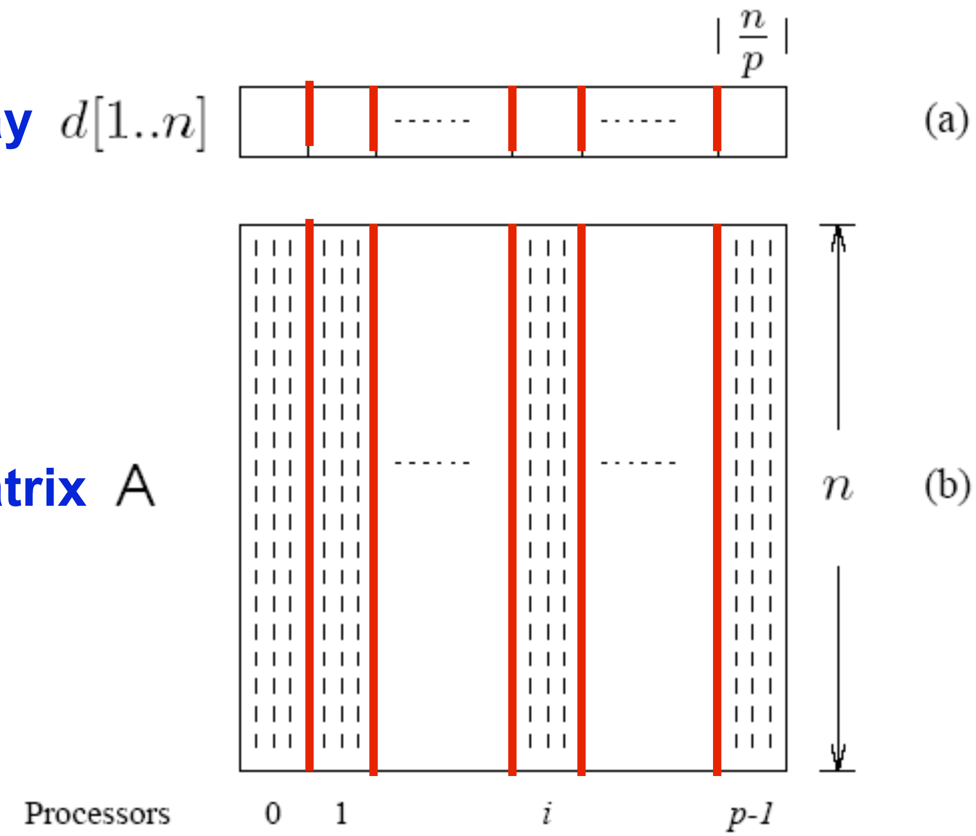
Parallel Formulation of Prim's Algorithm

- **Parallelization prospects**
  - —outer loop (|V| iterations): hard to parallelize
    - adding 2 vertices to tree concurrently is problematic
  - —inner loop: relatively easy to parallelize
    - consider which vertex is closest to MST in parallel
- **Approach**
  - —data partitioning
    - partition adjacency matrix in a 1-D block fashion (blocks of columns)
    - partition distance vector d accordingly
  - —in each step,
    - process first identifies the locally closest node
    - performs a global reduction to select globally closest node
    - leader inserts node into MST
    - broadcasts choice to all processes
    - each process updates its part of d vector locally based on choice

Parallel Formulation of Prim's Algorithm

distance array $d[1..n]$

adjacency matrix A



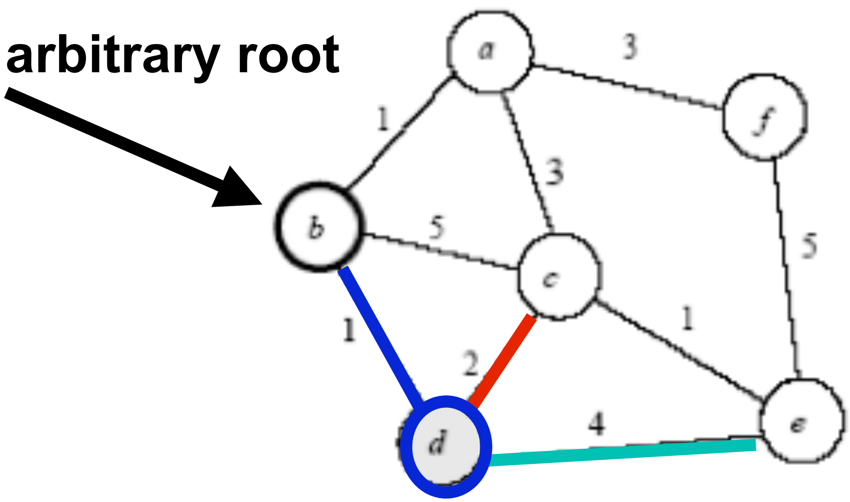
partition *d* and *A* among *p* processes

Parallel Formulation of Prim's Algorithm

- **Cost to select the minimum entry**
  - — *$O(n/p)$: scan n/p local part of d vector on each processor*
  - —*$O(\log p)$* all-to-one reduction across processors
- **Broadcast next node selected for membership**
  - —*$O(\log p)$*
- **Cost of locally updating *d* vector**
  - — *$O(n/p)$: replace d vector with min of d vector and matrix row*
- **Parallel time per iteration**
  - —*$O(n/p + \log p)$*
- **Total parallel time**
  - —*$O(n^2/p + n \log p)$*

Minimum Spanning Tree: Prim's Algorithm

start with arbitrary root



a	0	1	3	∞	∞	3
b	1	0	5	1	∞	∞
c	3	5	0	2	1	∞
d	∞	1	2	0	4	∞
e	∞	∞	1	4	0	5
f	2	∞	∞	∞	5	0

2 iterations of Prim’s algorithm

Algorithms for Sparse Graphs

- Dense algorithms can be improved significantly if we make use of the sparseness
- Example: Prim's algorithm complexity

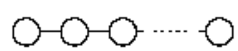
  —can be reduced to O(|E| log n)

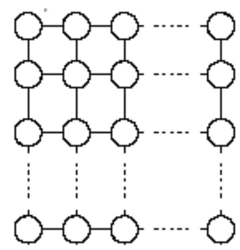
  – use heap to maintain costs
  – outperforms original as long as $|E| = O(n^2 / \log n)$
- Sparse algorithms: use adjacency list instead of matrix
- Partitioning adjacency lists is more difficult for sparse graphs

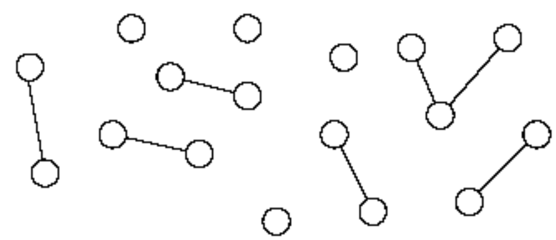
  — do we balance number of vertices or edges?
- Parallel algorithms typically make use of graph structure or degree information for performance

Algorithms for Sparse Graphs

Graph $G = (V,E)$ is sparse if $|E|$ is much smaller than $|V|^2$


(a)


(b)


(c)

Examples of sparse graphs: (a) a linear graph, in which each vertex has two incident edges; (b) a grid graph, in which each vertex has four incident vertices; and (c) a random sparse graph.

Topics for Today

- **Terminology and graph representations**
- **Minimum spanning tree, Prim's algorithm**
- **Shortest path, Dijkstra’s algorithm, Johnson’s algorithm**
- **Connected components**

Single-Source Shortest Paths

- Given weighted graph $G = (V,E,w)$
- Problem: *single-source shortest paths*
  - —find the shortest paths from vertex $v \in V$ to all other vertices in V
- Dijkstra's algorithm: similar to Prim's algorithm
  - —maintains a set of nodes for which the shortest paths are known
  - —grows set by adding node closest to source using one of the nodes in the current shortest path set

Computing Single-Source Shortest Paths

Dijkstra's sequential algorithm

```
procedure DIJKSTRA_SINGLE_SOURCE_SP(V, E, w, s)
begin
    V_T := {s};                // initialize tree vertices V_T with vertex s, the designated src
    for all v ∈ (V − V_T) do                          // compute l[:], the
        if (s, v) exists set l[v] := w(s, v);         // weight between
        else set l[v] := ∞;                           // s and each vertex ∉ V_T
    while V_T ≠ V do // while some vertices are not in V_T
    begin
        find a vertex u such that l[u] := min{l[v]|v ∈ (V − V_T)};
                                                      // use l[:] to find u,
                                                      // next vertex closest
                                                      // src
        V_T := V_T ∪ {u};  // add u to T
        for all v ∈ (V − V_T) do                      // recompute l[:]
            l[v] := min{l[v], l[u] + w(u, v)};        // now that u is in T
    endwhile
end DIJKSTRA_SINGLE_SOURCE_SP
```

Parallel Formulation of Dijkstra's Algorithm

Similar to parallel formulation of Prim's algorithm for MST

- **Approach**
  - —data partitioning
    - – partition weighted adjacency matrix in a 1-D block fashion
    - – partition distance vector L accordingly
  - —in each step,
    - – each process identifies its node closest to source
    - – perform a global reduction to select globally closest node
    - – broadcasts choice broadcast to all processes
    - – each process updates its part of L vector locally
- **Parallel performance of Dijkstra's algorithm**
  - — identical to that of Prim's algorithm
    - – parallel time per iteration: $O(n/p + \log p)$
    - – total parallel time: $O(n^2/p + n \log p)$

All-Pairs Shortest Paths

- **Given weighted graph *G(V,E,w)***
- **Problem: *all-pairs shortest paths***
  - — **find the shortest paths between all pairs of vertices v_i, $v_j \in V$**
- **Several algorithms known for solving this problem**

All-Pairs Shortest Path

Serial formulation using Dijkstra's algorithm

- **Execute *n* instances of the single-source shortest path**
  —one for each of the *n* source vertices
- **Sequential time per source: $O(n^2)$**
- **Total sequential time complexity: $O(n^3)$**

All-Pairs Shortest Path

Parallel formulation using Dijkstra's algorithm

Two possible parallelization strategies

- **Source partitioned: execute each of the *n* shortest path problems on a different processor**
- **Source parallel: use a parallel formulation of the shortest path problem to increase concurrency**

All-Pairs Shortest Path Dijkstra's Algorithm

"Source partitioned" parallel formulation

- **Use *n* processors**
  - —each processor P_i finds the shortest paths from vertex v_i to all other vertices
    - – use Dijkstra's sequential single-source shortest paths algorithm
- **Analysis**
  - —requires no interprocess communication
    - – provided adjacency matrix is replicated at all processes
  - —parallel run time: $\Theta(n^2)$
- **Algorithm is cost optimal**
  - —asymptotically same # of ops in parallel as in sequential version
- **However: can only use *n* processors (one per source)**

All-Pairs Shortest Path Dijkstra's Algorithm

"Source parallel" parallel formulation

- **Each of the shortest path problems is executed in parallel**
  - —**can therefore use up to n^2 processors.**
- **Given p processors (p > n)**
  - —**each single source shortest path problem is executed by p/n processors.**
- **Recall time for solving one instance of all-pair shortest path**
  - —***$O(n^2/p + n \log p)$***
- **Considering the time to do one instance on p/n processors**

$$T_P = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta(n \log p)}^{\text{communication}}.$$

- **Represents total time since each instance is solved in parallel**

Single-Source Shortest Paths

- **Dijkstra's algorithm, modified to handle sparse graphs is called Johnson's algorithm.**
- **The modification accounts for the fact that the minimization step in Dijkstra's algorithm needs to be performed only for those nodes adjacent to the previously selected nodes.**
- **Johnson's algorithm uses a priority queue *Q* to store the value $l[v]$ for each vertex $v \in (V - V_T)$.**

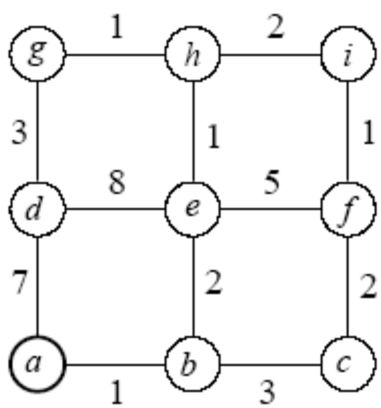
Single-Source Shortest Paths: Johnson's Algorithm

```
procedure JOHNSON_SINGLE_SOURCE_SP(V, E, s)
begin
    Q := V;
    for all v ∈ Q do
        l[v] := ∞;
    l[s] := 0;
    while Q ≠ ∅ do
    begin
        u := extract_min(Q);
        for each v ∈ Adj[u] do
            if v ∈ Q and l[u] + w(u, v) < l[v] then
                l[v] := l[u] + w(u, v);
    endwhile
end JOHNSON_SINGLE_SOURCE_SP
```

Single-Source Shortest Paths: Parallel Johnson's Algorithm

- **Maintaining strict order of Johnson's algorithm generally leads to a very restrictive class of parallel algorithms.**
- **We need to allow exploration of multiple nodes concurrently. This is done by simultaneously extracting *p* nodes from the priority queue, updating the neighbors' cost, and augmenting the shortest path.**
- **If an error is made, it can be discovered (as a shorter path) and the node can be reinserted with this shorter path.**

Single-Source Shortest Paths: Parallel Johnson's Algorithm



	Priority Queue	a	b	c	d	e	f	g	h	i
(1)	*b:1, d:7, c:inf, e:inf, f:inf, g:inf, h:inf, i:inf*	0	1	∞	7	∞	∞	∞	∞	∞
(2)	*e:3, c:4, g:10, f:inf, h:inf, i:inf*	0	1	4	7	3	∞	10	∞	∞
(3)	*h:4, f:6, i:inf*	0	1	4	7	3	6	10	4	∞
(4)	*g:5, i:6*	0	1	4	7	3	6	5	4	6

Array l[]

Single-Source Shortest Paths: Parallel Johnson's Algorithm

- **Even if we can extract and process multiple nodes from the queue, the queue itself is a major bottleneck.**
- **For this reason, we use multiple queues, one for each processor. Each processor builds its priority queue only using its own vertices.**
- **When process P_i extracts the vertex $u \in V_i$, it sends a message to processes that store vertices adjacent to *u*.**
- **Process P_j, upon receiving this message, sets the value of $l[v]$ stored in its priority queue to $min\{l[v],l[u] + w(u,v)\}$.**

Single-Source Shortest Paths: Parallel Johnson's Algorithm

- **If a shorter path has been discovered to node *v*, it is reinserted back into the local priority queue.**
- **The algorithm terminates only when all the queues become empty.**
- **A number of node paritioning schemes can be used to exploit graph structure for performance.**

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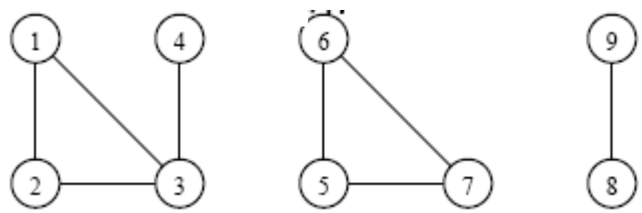
Connected Components

Definition: equivalence classes of vertices under the "is reachable from" relation for undirected graphs

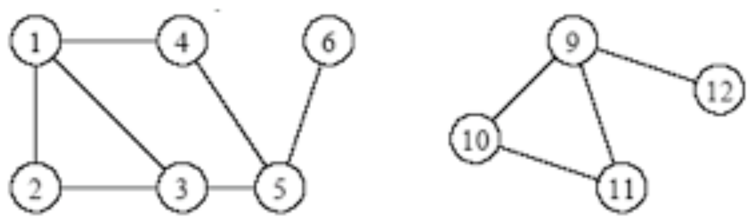


Example: graph with three connected components

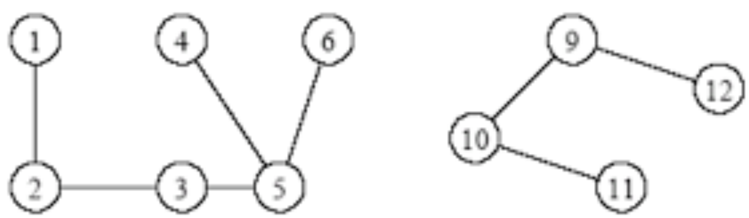
{1,2,3,4}*, *{5,6,7}*, and *{8,9}

Connected Components

Serial depth-first search based algorithm



- **Perform DFS on a graph to get a forest**
  —**each tree in the forest = separate connected component.**



Depth-first forest above obtained from depth-first traversal of the graph at top. result = 2 connected components

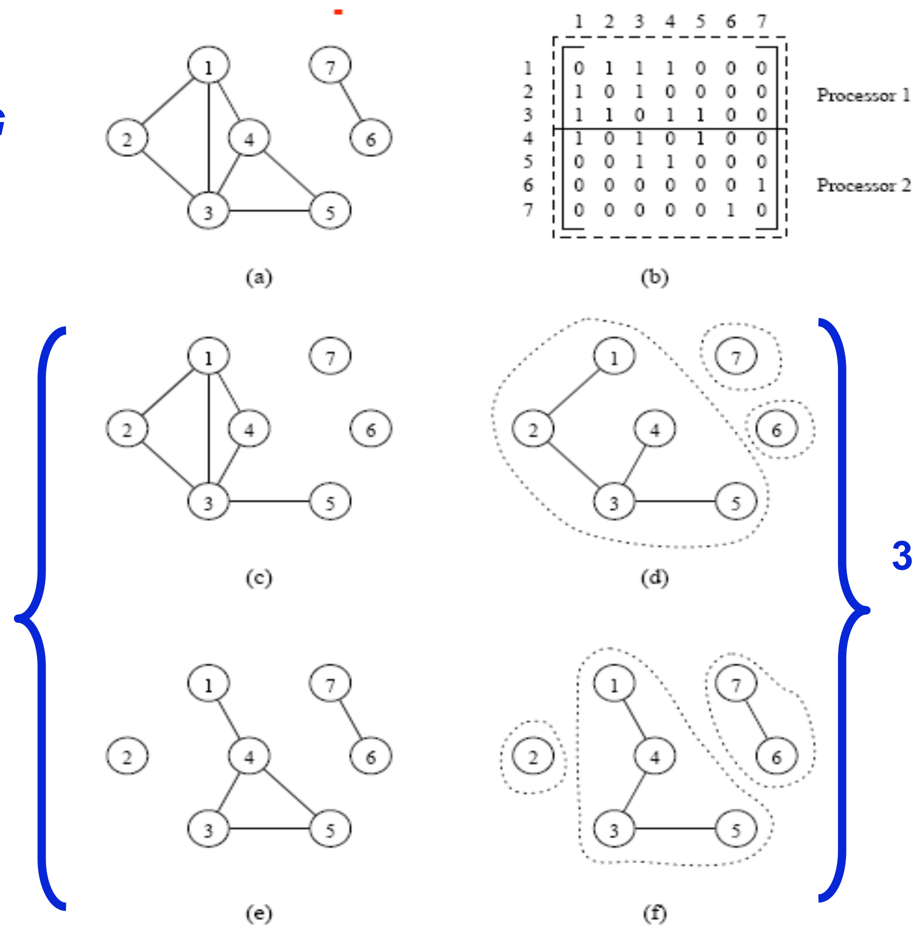
Connected Components Parallel Formulation

- **Partition the graph across processors**
- **Step 1**
  - **—run independent connected component algorithms on each processor**
  - **—result: *p* spanning forests.**
- **Step 2**
  - **—merge spanning forests pairwise until only one remains**

Connected Components Parallel Formulation

Graph *G*

1. Partition adjacency matrix of the graph *G* into two parts



2. Each process gets a subgraph of graph *G*

3. Each process computes the spanning forest of its subgraph of *G*

4. Merge the two spanning trees to form the final solution

Connected Components Parallel Formulation

- **Merge pairs of spanning forests using disjoint sets of vertices**
- **Consider the following operations on the disjoint sets**
  - *find*(*x*)
    - **returns pointer to representative element of the set containing *x***
    - **each set has its own unique representative**
  - *union*(*x, y*)
    - **merges the sets containing the elements *x* and *y***
    - **the sets are assumed disjoint prior to the operation**

Connected Components Parallel Formulation

- **To merge forest *A* into forest *B***
  - **—for each edge *(u,v)* of *A*,**
    - **perform *find* operations on u and v**
      - **determine if u and v are in same tree of *B***
    - **if not, then union the two trees (sets) of *B* containing *u* and *v***
      - **result: u and v are in same set, which means they are connected**
    - **else , no *union* operation is necessary.**
- **Merging forest *A* and forest *B* requires at most**
  - —*2(n-1)* ***find* operations**
  - —*(n-1)* ***union* operations**

} at most n-1 edges must be considered because A and B are forests

Connected Components

Analysis of parallel 1-D block mapping

- Partition an *n x n* adjacency matrix into *p* blocks
- Each processor computes local spanning forest: $\Theta(n^2/p)$
- Merging approach
  - —embed a logical tree into the topology
    - – *log p* merging stages
    - – each merge stage takes time $\Theta(n)$
  - —total cost due to merging is $\Theta(n \log p)$
- During each merging stage
  - —spanning forests are sent between nearest neighbors
  - —$\Theta(n)$ edges of the spanning forest are transmitted
- Parallel execution time

$$T_P = \overbrace{\Theta\left(\frac{n^2}{p}\right)}^{\text{local computation}} + \overbrace{\Theta(n \log p)}^{\text{forest merging}}.$$

Summary

- **Terminology and graph representations**
- **Minimum spanning tree, Prim's algorithm**
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